ISRAEL JOURNAL OF MATHEMATICS, Vol. 65, No. 2, 1989

"GAP 1" TWO-CARDINAL PRINCIPLES AND THE OMITTING TYPES THEOREM FOR $\mathscr{L}(Q)$†

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ABSTRACT

For λ a strong limit singular cardinal, and more generally for $\lambda > 2^{\mathrm{cof}\lambda}$, we prove the equivalence of a number of model theoretic and combinatorial conditions, including the $\mathscr{L}(Q)$-completeness theorem for the λ^+-interpretation, an omitting types theorem for $\mathscr{L}(Q)$ in the λ^+-interpretation, and a weak form of Jensen's principle $\square_\lambda$.

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§1. Idle chatter

1.1. *History*

For χ, λ, μ cardinals, we write

$$\langle \lambda, \lambda^+ \rangle \xrightarrow[\chi]{} \langle \mu, \mu^+ \rangle$$

to mean: for every first order theory T of cardinality less than χ with a

† Research supported in part by the United States–Israel Binational Science Foundation. The author thanks the referee for a thorough revision of the paper. Publication #269.

Received March 18, 1987 and in final revised form November 1, 1988

distinguished monadic predicate P, if every finite subset of T has a model of type (λ, λ^+) (that is, a model M of cardinality λ^+ in which P^M has cardinality λ), then T has a model of type (μ, μ^+). We will be interested here mainly in the case $\chi = \aleph_0$, i.e., T may be taken to be a single sentence; in this case we omit the reference to χ.

The subject began with the theorem of Vaught $(\langle \mu, \mu^+ \rangle \underset{\aleph_1}{\rightarrow} \langle \aleph_0, \aleph_1 \rangle)$. Subsequently Chang proved $(\langle \aleph_0, \aleph_1 \rangle \underset{\mu^+}{\rightarrow} \langle \mu, \mu^+ \rangle$ if $\mu = \mu^{<\mu})$. Jensen proved $(\langle \aleph_0, \aleph_1 \rangle \underset{\mu^+}{\rightarrow} \langle \mu, \mu^+ \rangle$ for μ strong limit singular) from $V = L$, more precisely $\square_\mu$; the published proof is due to Silver. In the negative direction, Silver and Mitchell proved Con$(\langle \aleph_0, \aleph_1 \rangle \not\rightarrow \langle \aleph_1, \aleph_2 \rangle)$, Schmerl showed (both using mild large cardinals) Con$(\forall n [\langle \aleph_n, \aleph_{n+1} \rangle \not\rightarrow \langle \aleph_{n+1}, \aleph_{n+2} \rangle])$, and Litman and Shelah showed

$$\mathrm{Con}(\mathrm{GCH} + [\langle \aleph_0, \aleph_1 \rangle \not\rightarrow \langle \aleph_\omega, \aleph_{\omega+1} \rangle])$$

(starting with supercompact cardinals). One of our goals here is to give precise set-theoretic equivalents to two-cardinal transfer principles, for strong limit singular cardinals.

Another goal is to settle the relationship of two-cardinal transfer principles to omitting types theorems for $\mathscr{L}(Q)$. Fuhrken showed the equivalence of the principle $\langle \aleph_0, \aleph_1 \rangle \underset{\aleph_0}{\rightarrow} \langle \mu, \mu^+ \rangle$ with a completeness theorem for $\mathscr{L}(Q)$ in the μ^+-interpretation (Qx being read as "for at least μ^+ x's"): "If $T \subseteq \mathscr{L}(Q)$ is consistent in the $\aleph_1$-interpretation, and $|T| \leqq \mu$, then T has a model for the μ^+-interpretation." Shelah [Sh1] showed that $\langle \aleph_0, \aleph_1 \rangle \underset{\aleph_1}{\rightarrow} \langle \mu, \mu^+ \rangle$ implies that $\mathscr{L}(Q)$ is μ-compact in the μ^+-interpretation, via a characterization of the transfer principle in terms of partition relations.†

Keisler proved a strong omitting types theorem for $\mathscr{L}(Q)$. One useful feature of his proof is that the final model is built as the union of a ω_1-chain of countable structures, which allows certain decisions about omission or realization of types to be made along the way; this flexibility is useful, e.g., Magidor and Malitz used it in the compactness of $\mathscr{L}(Q^n \cdots)_{n<\omega}$. This strengthening will be called the strong omitting types theorem. Keisler's result extends both Vaught's theorem and the Henkin omitting types theorem, and it is natural to look for common extensions to (μ-compact) models for the μ^+-interpretation for arbitrary cardinality μ.

† By [Sh3] even $\langle \aleph_0, \aleph_1 \rangle \rightarrow \langle \mu, \mu^+ \rangle$ implies this (as VIII of 2.2 can be axiomatized by $\psi \in L(Q)$).

Shelah [Sh4] derived a strong omitting types theorem for $\mathscr{L}(Q)$ in the λ^+-interpretation (i.e., with a "linear" proof, as in Keisler's case) from a diamond-like principle $(\mathrm{Dl})_\lambda$; one has the implications $\diamondsuit_\lambda \Rightarrow (\mathrm{Dl})_\lambda \Rightarrow \lambda = \lambda^{<\lambda}$, and in addition (by theorems of Gregory and Shelah, see [Sh4]) under GCH every regular $\lambda > \aleph_1$ satisfies $(\mathrm{Dl})_\lambda$ (and even $\diamondsuit_\lambda$ for λ a successor).

Grossberg [Gr] proved an omitting types theorem for $\mathscr{L}(Q)$ in the λ^+-interpretation for λ singular, assuming $\square_\lambda$ and $\{\chi < \lambda : (\mathrm{Dl})_\chi\}$ is unbounded in λ, using the arguments of Jensen or Silver (in $\langle \aleph_0, \aleph_1 \rangle \to \langle \aleph_\omega, \aleph_{\omega+1} \rangle$) and the arguments of [Sh4].

The question as to whether one can have a version strong enough for the construction of nearly rigid models, or transfer theorems for Magidor–Malitz quantifiers, [as the regular case was used] remains open.

On the relationship between the various relatives of the squares for successor of singular cardinals, see Ben David and Shelah [BSh].

1.2. *Notation*

For a logic $\mathscr{L}$ and a vocabulary τ, $\mathscr{L}[\tau]$ is the set of $\mathscr{L}$-formulas in this vocabulary. The logic $\mathscr{L}(Q)$ is the extension of first order logic by an additional quantifier Q, and in the λ-interpretation "Qx" signifies: "there are at least λ x such that". For an $\mathscr{L}(Q)$-theory T, consistency means finite consistency in the $\aleph_1$-interpretation, or equivalently: relative to Keisler's axioms.

Infinitary formulas Γ of the following form are taken to represent certain sets of $\mathscr{L}(Q)$-formulas, called their "finitary approximations":

$$(\Gamma) \qquad (Q_i \bar{y}_i)_{i<\alpha} \mathop{\&}_{j<\beta} \psi_j$$

where for $i < \alpha$, Q_i is $\exists$ or Q, and $\bar{y}_i$ is a sequence of variables, and for $j < \beta$ ψ_j is an $\mathscr{L}(Q)$-formula. A finitary approximation γ to such a formula Γ is a formula of the form:

$$(\gamma) \qquad (Q_{i(0)} \bar{y}^*_{i(0)} Q_{i(1)} \bar{y}^*_{i(1)} \cdots Q_{i(k)} \bar{y}^*_{i(k)}) \mathop{\&}_{j \in w} \psi_j$$

where $i(0) < i(1) < \cdots < i(k) < \alpha$, $\bar{y}^*_{i(l)} \subseteq \bar{y}_{i(l)}$ is a finite string of variables containing all free variables of any ψ_j $(j \in w)$ lying in $\bar{y}_{i(l)}$, and $w \subseteq \beta$ is also finite. Here the quantifier $Q\bar{y}$ is to be interpreted (for $\bar{y}$ finite) in the λ^+-interpretation as: "there are λ^+ *pairwise disjoint* sequences $\bar{y}_\alpha$ $(\alpha < \lambda^+)$ such that ...". So (Γ) represents the set of all of these finitary approximations (γ), and if Γ is already finitary then it is equivalent to its set of finitary approximations as defined here.

In connection with the omitting types property we will need the notion of λ-support from [Sh4], which is a natural extension of Keisler's notion. If T is an $\mathcal{L}(Q)$-theory, $p(\bar{x})$ a type, and Γ an infinitary formula of the form

$$(\Gamma) \qquad (Q_i \bar{y}_i)_{i<\alpha} \exists \bar{x} \mathop{\&}_{j<\delta} \psi_j(\bar{x}, \bar{y})$$

with all Q_i of the form $\exists$ or Q, we call Γ a λ-support for p if $\delta < \lambda$, Γ is consistent with T and for all $\varphi \in p$ the following is inconsistent with T:

$$(\Gamma(\varphi)) \qquad (Q_i \bar{y}_i)_{i<\alpha} \exists \bar{x} \left(\mathop{\&}_{j<\delta} \psi_j(\bar{x}, \bar{y}) \mathbin{\&} \neg \varphi(\bar{x}) \right).$$

REMARK. We can use only $Q_i y_i$ — see end of 3.1.

§2. A theorem or two

As far as the two-cardinal transfer principle $\langle \aleph_0, \aleph_1 \rangle \to \langle \lambda, \lambda^+ \rangle$ is concerned, if we assume GCH then only singular cardinals are problematic. Our main result clarifies this case:

2.1. THEOREM. *For λ a singular strong limit cardinal, and more generally if $\lambda > 2^{\operatorname{cof} \lambda}$, the following are equivalent:*

(1) $\langle \aleph_0, \aleph_1 \rangle \to \langle \lambda, \lambda^+ \rangle$.

(2) *The Completeness Theorem for $\mathcal{L}(Q)$ in the λ^+-interpretation.*

(3) *The omitting types theorem for $\mathcal{L}(Q)$ in the λ^+-interpretation.*

(4) *Various weak forms of $\square_\lambda$.*

A considerable amount remains to be filled in to make this precise. We will formulate eight properties of a cardinal λ, three model theoretic and the rest set-theoretic. A more precise version of 2.1 states that for singular strong limit cardinals they are all equivalent, and more generally that if $\lambda > 2^{\operatorname{cof} \lambda}$ then six of them are equivalent (we lose two versions of square to the vagaries of cardinal arithmetic).

2.2. *Eight properties of λ*

I. Two-cardinal transfer:

$$\langle \aleph_0, \aleph_1 \rangle \to \langle \lambda, \lambda^+ \rangle.$$

II. $\mathcal{L}(Q)$-completeness:

If T is a consistent $\mathscr{L}(Q)$-theory of cardinality at most λ, then T has a model in the λ^+-interpretation.

III. Omitting types:
If T is a consistent $\mathscr{L}(Q)$-theory of cardinality at most λ and p_i $(i<\lambda)$ are types with no λ-support, then T has a model in the λ^+-interpretation omitting each p_i.

The types p_i are assumed to involve finitely many free variables, or equivalently just one free variable. For types in κ variables our results will hold if λ satisfies additionally: $\operatorname{cof}\lambda>\kappa$, and $|\alpha^\kappa|<\lambda$ for $\alpha<\lambda$.

The next four properties are weak variants of Jensen's square (cf. 2.4). A sequence $\langle C_\alpha:\alpha<\lambda^+, \alpha \text{ a limit}\rangle$ of sets $C_\alpha\subseteq\alpha<\lambda^+$ is a $\square$ sequence on λ^+ (or: "for λ") if:

(1) C_α is closed unbounded in α for $\alpha<\lambda^+$ a limit;
(2) $|C_\alpha|<\lambda$ if $\operatorname{cof}(\alpha)<\lambda$ (always, if λ is singular);
(3) if $\alpha,\beta<\lambda^+$ are limits with $\beta\in C'_\alpha$ (a limit of elements of C_α) then $C_\beta=\beta\cap C_\alpha$.

Jensen's principle $\square_\lambda$ asserts that there is a $\square$ sequence on λ^+.

Terminology

We will consider sequences $\langle C_\alpha:\alpha<\lambda^+\rangle$ with $C_\alpha\subseteq\alpha$ for $\alpha<\lambda^+$. Such a sequence is *coherent* if $C_\beta=\beta\cap C_\alpha$ whenever $\beta\in C_\alpha$. A family $\bar{C}=\langle C^\zeta_\alpha:\alpha<\lambda^+, \zeta<\operatorname{cof}\lambda\rangle$ will be called a λ^+*-decomposition* if:

(1) for $\zeta<\operatorname{cof}\lambda$ the sequence $\langle C^\zeta_\alpha:\alpha<\lambda^+\rangle$ is coherent;
(2) for $\alpha<\lambda^+$, $\alpha=\bigcup_\zeta C^\zeta_\alpha$ (an increasing union in ζ).

We associate with any λ^+-decomposition $\bar{C}$ the structures $\mathscr{C}^\zeta_\alpha$ $(\alpha<\lambda^+, \zeta<\operatorname{cof}\lambda)$ with underlying set $\{\alpha\}\cup C^\zeta_\alpha$ and relations $<$ and R^ξ (for $\xi<\zeta$), where $R^\xi(i,j)$ signifies: $i\in C^\xi_j$.

IV. $\square^a_\lambda$:
There is a λ^+-decomposition $\bar{C}$ such that for any $\zeta<\operatorname{cof}\lambda$: there are fewer than λ associated structures of the form $\mathscr{C}^\zeta_\alpha$ $(\alpha<\lambda^+)$, up to isomorphism.[†]

[†] Note that this implies $|C^\zeta_\alpha|<\lambda$, as $\langle\mathscr{C}^\zeta_\beta:\beta\in C^\zeta_\alpha\rangle$ are pairwise non-isomorphic having different order type.

V. $\square_\lambda^b$:
There is a λ^+-decomposition $\bar{C}$ such that for any $\zeta < \operatorname{cof}\lambda$ there are at most λ associated structures of the form $\mathscr{C}_\alpha^\zeta$ ($\alpha < \lambda^+$), up to isomorphism.

We will need a weak form of the last two principles. Observe that in IV and V λ is involved as a cardinality, but λ^+ enters primarily as an ordered set. If L is a λ^+-like ordering (i.e. $|L| = \lambda^+$, each proper initial segment is of smaller cardinality) then we may speak of L-decompositions $\langle C_\alpha^\zeta : \zeta < \operatorname{cof}\lambda,\ a \in L\rangle$ and associated structures $\mathscr{C}_\alpha^\zeta$, and then formulate the following variants of IV, V.

VI. $\square_\lambda^{a*}$:
There is a λ^+-like ordering L, an L-decomposition $\bar{C}$, and a sequence of refining equivalence relations E^ζ ($\zeta < \operatorname{cof}\lambda$) on L, such that for any $\zeta < \operatorname{cof}\lambda$ and $a, b \in L$:
(1) $E^\zeta(a, b)$ implies that there is an isomorphism $h : \mathscr{C}_a^\zeta \simeq \mathscr{C}_b^\zeta$ such that for $a' \in \mathscr{C}_a^\zeta$, $E^\zeta(a', h(a'))$ holds.
(2) $a \in \mathscr{C}_b^\zeta$ implies $\neg E^\zeta(a, b)$.
(3) E^ζ has fewer than λ equivalence classes.

VII. $\square_\lambda^{b*}$:
There is a λ^+-ordering L, an L-decomposition $\bar{C}$, and a sequence of refining equivalence relations E^ζ on L, such that for any $\zeta < \operatorname{cof}\lambda$ and $a, b \in L$ conditions (1–2) of $\square_\lambda^{a*}$ hold, and:
(3′) E^ζ has at most λ equivalence classes.

The last property that we consider is quoted from [Sh1] (for the general form) and more specifically from [Sh3].

VIII. A non-partition property:
There is a λ-coloring $c : (\lambda^+)^2 \to \lambda$ such that for any finite $w \subseteq \lambda^+$ there are finite sets w_0, w_1, w_2 such that for $\alpha_i \in w_i$ ($i = 0, 1, 2$) we have $\alpha_0 < \alpha_1 < \alpha_2$, and there are color- and order-preserving functions $f : w \xrightarrow{\text{onto}} w_0 \cup w_1 \cup w_2$ and $g : w_0 \cup w_1 \to w_0 \cup w_2$ (e.g. $c(\alpha, \beta) = c(f(\alpha), f(\beta))$ on w^2) so that:
$f[w]$ meets both w_1 and w_2;
$c[w_1 \times w_2]$ is disjoint from $c[(w_0 \cup w_i)^2]$ for $i = 1, 2$; and
$c \restriction (w_1 \times w_2)$ is one-to-one.

2.3. Main Theorem. *If λ is a singular strong limit cardinal then these eight properties are equivalent. If $\lambda > 2^{\operatorname{cof}\lambda}$, properties* I–IV, VI, VIII *are equivalent. More precisely, for any λ:*
(1) I, II, VII, VIII *are equivalent,*

(2) III $\Rightarrow$ II, IV $\Rightarrow$ VI, VI $\Rightarrow$ VII, V $\Rightarrow$ VII, *and* IV $\Rightarrow$ V,
(3) II $\Rightarrow$ VI *if* $2^{\operatorname{cof}\lambda} < \lambda$,
(4) VI $\Rightarrow$ III,
(5) VI $\Rightarrow$ IV *if* λ *is a strong limit cardinal.*

PROOF. The proof is arranged as follows.
(1) That I, II are equivalent is due to Fuhrken.
That I, VIII are equivalent is found in [Sh3].
I $\Rightarrow$ VII: §5.1.
VII $\Rightarrow$ II: §4.
(2) These are all clear.
(3) §5.2.
(4) §4.
(5) §2.5.

2.4. *Other relationships*

The following results are not needed for the proof of the main theorem, but serve to clarify various relationships among combinatorial notions.
(1) If $\lambda^{<\lambda} = \lambda$ then $\square_\lambda^b$ holds, and if λ is strongly inaccessible then $\square_\lambda^a$ holds (2.7).
(2) $\square_\lambda$ implies $\square_\lambda^a$ for λ a singular strong limit (2.8).
(3) $\square_\lambda^a$ yields a λ-Kurepa tree (2.6). (So if e.g. GCH holds and λ is regular, then the condition $\square_\lambda^a$ can fail.)

2.5. REMARK. Assume $\square_\lambda^a$. Then for $\alpha < \lambda^+$, $\zeta < \operatorname{cof}\lambda$, and $\beta, \gamma \in \{\alpha\} \cup C_\alpha$ distinct, C_β^ζ is not isomorphic with C_γ^ζ, since if e.g. $\beta < \gamma$ then $\operatorname{ot}(C_\beta^\zeta) < \operatorname{ot}(C_\gamma^\zeta)$. Hence if there are κ isomorphism types of structures among $\{C_\alpha^\zeta : \alpha < \lambda^+\}$, then $\operatorname{ot}(C_\alpha^\zeta) < \kappa^+$ for each such structure (even uniformly in α). In other words, if $\bar{C}$ witnesses $\square_\lambda^a$ then for each $\zeta < \operatorname{cof}\lambda$ we will have:

$$(*) \qquad \sup\{\operatorname{ot}(C_\alpha^\zeta) : \alpha < \lambda^+\} < \lambda$$

if λ is a limit cardinal. Conversely if λ is a strong limit cardinal and $\bar{C}$ is a λ^+-decomposition, then the latter condition suffices for $\square_\lambda^a$.

In particular, when λ is a strong limit cardinal we can derive $\square_\lambda^a$ from $\square_\lambda^{a*}$ by restricting an L-decomposition $\bar{C}$ to a cofinal λ^+-sequence in L, since $(*)$ also follows from $\square_\lambda^{a*}$.

2.6. FACT. $\square_\lambda^a$ yields a λ-Kurepa tree, that is a tree with $\operatorname{cof}(\lambda)$ levels, each of size less than λ, and at least λ^+ branches.

PROOF. Let $\langle C_\alpha^\zeta : \alpha < \lambda^+, \zeta < \operatorname{cof} \lambda \rangle$ be a λ^+-decomposition afforded by $\Box_\lambda^a$. For any associated structure $\mathscr{C}_\alpha^\zeta$ let M_α^ζ be the canonical collapse of C_α^ζ to a structure whose underlying set is an ordinal. The M_α^ζ for $\alpha < \lambda^+$ will be at level ζ in the tree; by $\Box_\lambda^a$, for each $\zeta < \operatorname{cof}(\lambda)$, there are fewer than λ such collapsed structures. Put M_α^ζ above M_β^ξ (for $\xi < \zeta$) if $\operatorname{ot}(M_\beta^\xi) \leqq \operatorname{ot}(M_\alpha^\zeta)$ and the canonical injection is an isomorphism for the language of M_β^ξ. Each ordinal $\alpha < \lambda^+$ (i.e., the identity) determines a branch $\langle M_\alpha^\zeta : \zeta < \operatorname{cof} \lambda \rangle$, and these branches are distinct, since for $\beta < \alpha$ and ζ large we will have $\beta \in C_\alpha^\zeta$ and hence $\{\beta\} \cup C_\beta \subseteq C_\alpha$, forcing $\operatorname{ot}(M_\alpha^\zeta) > \operatorname{ot}(M_\alpha^\zeta)$.

2.6a. NOTE. $\Box_\lambda^{a*}$ would suffice.

2.7. PROPOSITION. *If $\lambda^{<\lambda} = \lambda$ then $\Box_\lambda^b$ holds, while if λ is strongly inaccessible then $\Box_\lambda^a$ also holds.*

PROOF. We defer the case $\lambda = \aleph_0$ to stage A of the proof of 5.1 as some details are different, and we actually require $\Box_{\aleph_0}^a$ in the proof of the main result. So assume λ is uncountable. It will suffice to construct a λ^+-decomposition

$$\langle C_\alpha^\zeta : \zeta < \lambda, \alpha < \lambda^+ \rangle$$

so that $|C_\alpha^\zeta| \leqq |\zeta|$. We proceed by induction on α, and our inductive hypothesis includes the condition: $\beta = \bigcup_{\zeta < \lambda} C_\beta^\zeta$ for $\beta < \alpha$, which we therefore check as we proceed. There are four cases.

If $\alpha = 0$ set $C_\alpha^\zeta = \varnothing$.

If $\alpha = \delta + k$ with δ a limit ordinal and $k > 0$ an integer, we set $C_\alpha^\zeta = C_\delta^\zeta \cup (\alpha - \delta)$.

If $\alpha = \delta$ is a limit ordinal of cofinality κ we set $\delta = \lim_{i<k} \delta(i)$ (increasing) and deal with two cases. Suppose first that $\kappa < \lambda$. Then for some ξ with $\kappa \leqq \xi < \lambda$ we have:

$$\delta(i) \in C_{\delta(j)}^\xi \qquad \text{for all } i < j < \kappa.$$

Let $C_\alpha^\zeta = \varnothing$ for $\zeta \leqq \xi$, and $C_\alpha^\zeta = \bigcup_{i<k} C_{\delta(i)}^\zeta$ for $\zeta > \xi$.

Finally we suppose that $k = \lambda$ and we retain the notation of the previous case. For $\xi, \zeta < \lambda$ call ζ ξ-adequate if:

$$\delta(j) \in C_{\delta(i)}^\zeta \qquad \text{for } j < i < \xi.$$

Define $f: \lambda \to \lambda$ by $f(\zeta) = \sup\{\xi : \xi \leqq \zeta \text{ and } \zeta \text{ is } \xi\text{-adequate}\}$. Then f is monotonically nondecreasing and ζ is $f(\zeta)$-adequate for all $\zeta < \lambda$. For any $i < \lambda$ there is $\zeta < \lambda$ with $\zeta \geqq i$ and ζ i-adequate, hence $f(\zeta) \geqq i$. Now define $C_\alpha^\zeta = \bigcup_{i<f(\zeta)} C_{\delta(i)}^\zeta$. The coherence and cardinality constraints are respected,

and these sets increase as functions of ζ. As $\mathrm{rg}(f)$ is unbounded, $\alpha = \bigcup_\zeta C^\zeta_\alpha$ by induction. ■

2.8. PROPOSITION. *$\square_\lambda$ implies $\square^a_\lambda$ for λ a singular strong limit.*

PROOF. Let $\langle C_\alpha : \alpha < \lambda^+,\ \alpha \text{ a limit}\rangle$ be a $\square$ sequence for λ^+. Let $\lambda = \lim_{i<\mathrm{cof}\,\lambda} \lambda(i)$ with $\langle \lambda(i)\rangle_{i<\mathrm{cof}\,\lambda}$ an increasing sequence of cardinals. We will construct a λ^+-decomposition $\bar{C}$ satisfying:

(1) $|C^\zeta_\alpha| \leqq \lambda(\zeta)$,

(2) for α a limit, $\beta \in C'_\alpha$, and $\zeta < \mathrm{cof}\,\lambda$ with $\lambda(\zeta) \geqq |C_\alpha|$, we have $\beta \cap C^\zeta_\alpha = C^\zeta_\alpha$.

We proceed by induction on α.

$$C^\zeta_0 = \varnothing \qquad \text{for } \zeta < \mathrm{cof}\,\lambda.$$

If $\alpha = \delta + k$ with $0 < k < \omega$, δ a limit, let $C^\zeta_\alpha = C^\zeta_\alpha \cup (\alpha - \delta)$.

Suppose that α is a limit. Then C_α exists. There are two cases. First, if $\alpha \in C''_\alpha$ (equivalently, $\alpha = \sup(C'_\alpha)$) let C^ζ_α be:

$$\begin{array}{ll} \varnothing & \text{if } \lambda(\zeta) < |C_\alpha|; \\ \bigcup\{C^\zeta_\beta : \beta \in C'_\alpha, b < \alpha\} & \text{if } \lambda(\zeta) \geqq |C_\alpha|. \end{array}$$

The coherence condition follows in the last case, since for $\beta < \gamma < \alpha$ with $\beta, \gamma \in C'_\alpha$ we have: $\beta \in C'_\gamma$ and $|C_\gamma| \leqq \lambda(\zeta)$, so by (2) inductively $\beta \cap C^\zeta_\gamma = C^\zeta_\beta$.

Finally, we may suppose that α is a limit and $\alpha > \sup C'_\alpha = \beta$. So $C_\alpha - \beta$ has order type ω, say $C_\alpha - \beta = \langle \gamma(0), \gamma(1), \ldots\rangle$ in increasing order. Call ζ *n-adequate* if $\gamma(i) \in C^\zeta_{\gamma(j)}$ for $i < j \leqq n$. Let $C^\zeta_\alpha = \bigcup\{C^\zeta_{\gamma(i)} : \zeta \text{ is } i\text{-adequate}\}$. Note that for all ζ $C^\zeta_\beta = C^\zeta_{\gamma(0)} \subseteq C^\zeta_\alpha$, so if $\delta \in C'_\alpha$ and $\lambda(\zeta) \geqq |C_\alpha|$, then as $\delta \leqq \beta$ we find: $\delta \cap C^\zeta_\alpha = \delta \cap C^\zeta_\beta = C^\zeta_\delta$ by induction. ■

2.9. NOTE TO 2.8. If $W \subseteq V$ are models of set theory, λ and λ^{+^W} are cardinals in W, and in W $\square^a_\lambda$ holds, then in V $\square^a_\lambda$ also holds, so normally it is enough that λ be a singular cardinal.

2.10. REMARK. We could replace condition II by the following without modifying any of the arguments given:

II′. Any consistent $\mathscr{L}(Q)$ sentence has a model in the λ^+-interpretation.

As in 5.1, 5.2 we actually use II′ and not I as an assumption. This then yields another proof that II and II′ are equivalent.

§3. Arboriculture

3.1. *Introduction*

In the balance of this paper we will deal with the main issue: how to carry out a Henkin construction for $\mathscr{L}(Q)$ over a tree of approximations to a model of size λ^+, given a suitable $\square$-like principle as a point of departure. Here we focus on the necessary syntactical preliminaries concerning trees of types. It will be convenient to invoke Keisler's completeness theorem for $\mathscr{L}(Q)$ in the $\aleph_1$-interpretation. Thus it suffices to check the correctness of certain arguments in this interpretation, rather than providing an explicit formal derivation of the necessary principles in an axiomatic framework.

As a matter of notation we introduce additional quantifiers $Q_n x_1 \cdots x_n$, where $Q_n \bar{x}\varphi$ means: there are at least λ^+ *disjoint* sequences $x_1, \dots, x_n$ satisfying φ. These quantifiers may be defined inductively in $\mathscr{L}(Q)$ as follows:

$$Q_{n+1}\bar{x}y\,\varphi \equiv:$$

$$(QyQ_n\bar{x}\varphi) \vee (Q_n\bar{x}\ \exists y[\varphi \,\&\, \neg Q\bar{x}'\varphi(\bar{x}'y)]).$$

Again, the basic properties of these quantifiers can be ascertained by inspecting the $\aleph_1$-interpretation.

3.2. *Trees of types*

Let $\mathscr{T}$ be a tree, that is a partial order with unique minimum 0, such that for $t \in T$, $\{s \in T : s < t\}$ is linearly ordered. A $\mathscr{T}$-tree of types is an assignment $\bar{z}_t$, p_t of variables $\bar{z}_t$ (possibly infinitely many) and partial types p_t to the nodes t of $\mathscr{T}$, so that:

(1) the sequences $\bar{z}_t$ ($t \in \mathscr{T}$) are pairwise disjoint, and each is partitioned into two strings $\bar{z}_t = \bar{x}_t ; \bar{y}_t$,

(2) p_t is a consistent type in the variables $Z_t = \bigcup \{\bar{z}_s : s \leqq t\}$,

(3) for $s < t$, $p_s = p_t \upharpoonright Z_s$ (more accurately: $p_s \subseteq p_t$ and $p_s \vdash p_t \upharpoonright Z_s$),

(4) for $s < t$ all finite approximations to $(\cdots Q\bar{x}_u\ \exists \bar{y}_u \cdots)_{s<u\leqq t} \wedge p_t$ belong to p_s,

(5) $\bar{z}_0 = \varnothing$.

In (4) the variables $\bar{x}_u$, $\bar{y}_u$ occur in the order of increasing u, and $Q\bar{x}_u$ is to stand for $Q_n\bar{x}'_u$ in any finite approximation for which $\bar{x}'_u \subseteq \bar{x}_u$ has length n. The variables $\bar{x}_u$, $\bar{y}_u$ for $u \leqq s$ occur freely in (4). In terms of the idea as sketched above, the variables $\bar{x}_t$, $\bar{y}_t$ are witnesses for Q and $\exists$ respectively.

If T is an $\mathscr{L}(Q)$-theory then $\bar{p}$ will be called a $(T, \mathscr{T})$-tree of types if, in addition, each p_t is consistent with T. We will say that $\bar{p}$ is deductively closed (that is, within its own vocabulary) if each p_t is (relative to the full vocabulary of $\bar{p}$). Given two $\mathscr{T}$-trees $\bar{p}$, $\bar{q}$ we will say that $\bar{p} \subseteq \bar{q}$ if $p_t \subseteq q_t$ for all $t \in \mathscr{T}$. $|\bar{p}|$ denotes the maximum of $|\mathscr{T}|$, $\sup\{|p_t| : t \in \mathscr{T}\}$, and $\aleph_0$. In our application we ignore the case $\lambda = \aleph_0$, which in any case was handled by Keisler, since it would require minor terminological modifications.

3.3. Remark. If $\bar{p}$ is a tree of types and $\bar{q}$ is its deductive closure (i.e. each q_s is the deductive closure of p_s; namely, we put φ in q_s if $p_s \vdash \varphi$ and the set of free variables of φ is $\subseteq \bar{z}_s$; we shall usually ignore such points) then $\bar{q}$ is again a tree of types.

3.4. Extension Lemma. *Let $\bar{p}$ be a $\mathscr{T}$-tree of types and let $t \in \mathscr{T}$. Suppose $p^* \supseteq p_t$ is a consistent type and p^* contains all finite approximations to*:

$$(*) \qquad (\cdots Q\bar{x}_u \ \exists \bar{y}_u \cdots)_{s<u\leqq t} \wedge p^*$$

for $s \leqq t$.

Then there is a $\mathscr{T}$-tree $\bar{q}$ of types with $\bar{p} \subseteq \bar{q}$, $p^* \subseteq q_t$, and $|\bar{q}| = |\bar{p}| + |p^*|$.

Proof. We may take $\bar{p}$ and p^* to be deductively closed. For $t' \in \mathscr{T}$ define $p^*_{t'} = p^* \restriction \bigcup \{Z_s : s \leqq t, t'\}$, and let $q_{t'}$ be the union of the sets of finite approximations to:

$$(\dagger) \qquad (\cdots Q\bar{x}_u \ \exists \bar{y}_u \cdots)_{s<u\leqq t'} \wedge p^*_{t'} \,\&\, \wedge p_{t'}$$

for $s \leqq t'$, $s \in \mathscr{T}^+$.

For $t'' \leqq t'$ in $\mathscr{T}$ certainly $q_{t''} \subseteq q_{t'}$. We claim also that $q_{t''} \vdash q_{t'} \restriction Z_{t''}$. Fix $s \leqq t''$ and consider a finite approximation Φ to ($\dagger$) with matrix $\varphi = \varphi_1 \,\&\, \varphi_2$, where $\varphi_1 \in \wedge p^*_{t'}$ and $\varphi_2 \in \wedge p_{t'}$. Let

$$s' =: \max(s, \sup\{s_0 \leqq t : \varphi_1 \text{ has variables in } Z_{s_0}\}).$$

Let $\varphi' = (\cdots Q\bar{x}_u \ \exists \bar{y}_u \cdots)_{s'<u\leqq t'} \varphi$. Then $\varphi' \equiv \varphi_1 \,\&\, \varphi_2'$ with $\varphi_2' = (\cdots Q\bar{x}_u \ \exists \bar{y}_n \cdots)_{s'<u\leqq t'} \varphi_2$ and (essentially) Φ is $(\cdots Q\bar{x}_u \ \exists \bar{y}_u \cdots)_{s<u\leqq s'} \varphi'$.

If $s' \leqq t$ then $\varphi_2' \in p_{s'} \subseteq p^*$, so φ and hence Φ are in p^*. Thus $\Phi \in p^* \restriction Z_s \subseteq q_{t''}$. If $s' \not\leq t$ then $s' = s$ and $\Phi = \varphi_1 \,\&\, \varphi_2'$ with $\varphi_1 \in p^*_{s'} \subseteq q_{t''}$, $\varphi_2' \in p_{s'} \subseteq q_{t''}$.

This proves (3) of the definition of a tree of types. To verify (2), consistency, in view of (3) it suffices to check the consistency of $q_{t'}$ which is p^*. Also (3) allows us to reduce (4) to:

(4) For $t' \in \mathscr{T}$, if $\varphi \in \wedge q_t$ then $q_t \vdash Q\bar{x}_t \ \exists \bar{y}_t \varphi_t$

which holds in our case. ■

3.5. LEMMA. *Let T be an $\mathscr{L}(Q)$-theory, p a partial $\mathscr{L}(Q)$-type in the variables $\bar{x}_i$, $\bar{y}_i$ ($i \in L$, a linear order), and assume that all finite approximations to*:

$$(*) \qquad [\cdots Q\bar{x}_i \ \exists \bar{y}_i \cdots]_{i\in L} \wedge p$$

are consistent with T. Let q be the type consisting of the union of all the sets of finite approximations to:

$$(\dagger)_p \qquad [\cdots Q\bar{x}_i \ \exists \bar{y}_i \cdots]_{i>s} \wedge p$$

as s varies over L. Then:

1. *$p \subseteq q$, q is consistent with T, and any finite approximation to $(\dagger)_q$ (putting q for p) is a consequence of q.*

2. *For any formula φ there is a choice of $\psi = \varphi$ or $\neg\varphi$ so that all finite approximations to*:

$$(*2) \qquad [\cdots Q\bar{x}_i \ \exists \bar{y}_i \cdots]_{i\in L} \wedge (p \cup \{\psi\})$$

are consistent with T.

3. *Let $0 \in L$ be minimal, $\varphi(\bar{x}, \bar{y})$ an $\mathscr{L}(Q)$-formula, $\bar{u}$, $\bar{z}$ variables with $\bar{u}$ disjoint from $\bar{x}_i$, $\bar{y}_i$ for $i \in L - \{0\}$, and $\bar{z}$ disjoint from all $\bar{x}_i$, $\bar{y}_i$, and $\bar{u}$, and set $\bar{y}'_i = \bar{y}_i$ for $i > 0$, $\bar{y}'_0 = \bar{y}_0 \cup \bar{z}$. Then*:

$$(*3) \qquad [\cdots Q\bar{x}_i \ \exists \bar{y}'_i \cdots]_{i\in L} \wedge (p \cup \{\exists \bar{x}\, \varphi(\bar{x}, \bar{u}) \rightarrow \varphi(\bar{z}, \bar{u})\})$$

is consistent with T.

4. *If $\varphi(\bar{x}, \bar{y})$, $\bar{u}$, $\bar{z}$ are as above, $\bar{u}$ disjoint to $\bar{y}_0$ and $\bar{x}'_i = \bar{x}_i$ for $i > 0$, $\bar{x}'_0 = \bar{x} \cup \bar{z}$, then*:

$$(*4) \qquad [\cdots Q\bar{x}'_i \ \exists \bar{y}_i \cdots]_{i\in L} \wedge (p \cup \{Q\bar{x}\, \varphi(\bar{x}, \bar{u}) \rightarrow \varphi(\bar{z}, \bar{u})\})$$

is consistent with T.

PROOF. This follows by inspecting the $\aleph_1$-interpretation. ■

3.6. COROLLARY (to 3.4, 3.5). *If $\bar{p}$ is a $\mathscr{T}$-tree of types then we can find a $\mathscr{T}$-tree $\bar{q}$ of types in the same language with $\bar{p} \subseteq \bar{q}$, $|\bar{q}| = |\bar{p}|$, each q_t complete (for the variables in Z_t) and such that*:

(a) *each q_t has existential witnesses*:
if $\varphi(\bar{u}, \bar{z})$ is an $\mathscr{L}(Q)$-formula with $\bar{z} \in Z_t$, then for some $\bar{y} \in \bigcup_{s \leqq t} \bar{y}_s$ $(\exists \bar{u}\, \varphi(\bar{u}, \bar{z}) \rightarrow \varphi(\bar{y}, \bar{z}))$ belongs to q_t;

(b) *each q_t has Q-witnesses*:
if $\varphi(\bar{u}, \bar{z})$ is an $\mathscr{L}(Q)$-formula with $\bar{z} \subseteq Z_s$, $s < s' \leqq t$, then for some $\bar{x} \subseteq \bar{x}_{s'}$, $[Q\bar{u}\, \varphi(\bar{u}, \bar{z}) \rightarrow \varphi(\bar{x}, \bar{z})]$ belongs to q_t;

(c) *for any* $s < t$ *with* $s > 0$, $\mathscr{L}(Q)$*-formula* $\varphi(\bar{z}, z)$, *and* $\bar{z} \in Z_s$, $z \subseteq \bar{z}_t$, *if* $\varphi(\bar{z}, z) \& \neg Qz\, \varphi(\bar{z}, z)$ *is in* p_t, *then for some* $z' \in \bar{y}_s$, $(z = z') \in p_t$.

PROOF. An iterative procedure based on Lemma 3.5 yields everything but condition (c).

To achieve (c) requires a further iteration based on Lemma 3.4. For this it is necessary to verify that under the hypothesis of (c) the type consisting of all finite approximations to expressions:

$$(*) \qquad [\cdots Q\bar{x}_n \exists \bar{y}_n \cdots]_{s' < u \leqq t}[\wedge p_t \& (z = z')]$$

$(s' < t)$ is consistent, where z' is a new variable adjoined to y_s (taken as bounded in $(*)$ if $s' < s$).

For $\varphi \in \wedge p_t$, $[\cdots Q\bar{x}_n \exists \bar{y}_n \cdots]_{s' < u \leqq t}\varphi$ is witnessed in a model M in the $\aleph_1$-interpretation by a certain tree of sequences in M, which must be thinned (if $s' < s$) so as to allow a corresponding choice of $z' = z$ along each branch. As there are at most $\aleph_0$ choices at the appropriate points, it is easy to thin this tree suitably. ■

3.7. REMARK. If the tree $\mathscr{T}$ is well-founded then (c) allows us to obtain a more extreme condition, assuming that for $t \in \mathscr{T} \setminus \{0\}$, $0 < h(t) \leqq t$, $h(t)$ an immediate successor of 0:

(c′) *for* $t \in \mathscr{T}$ *and* $y \in \bar{y}_t$, *there is* $z \in (\bigcup_{s \leqq t} \bar{x}_s) \cup \bar{z}_{h(t)}$ *such that* $(y = z) \in p_t$.

For this it suffices to check that if there are $s < t$, φ satisfying the hypothesis of (c), then $(z = x)$ may be added to p_t with x a new variable adjoined to $\bar{x}_t$. As we do not apply this stronger condition, we say no more about it.

§4. In memoriam Joyce Kilmer

4.1. *Introduction*

In what may be called the *linear* approach to building models of an $\mathscr{L}(Q)$-theory T in the λ^+-interpretation, we let M be $\bigcup \{M_\zeta : \zeta < \lambda^+\}$ with M_ζ a weak model of cardinality λ, taking care that in $M_{\zeta+1}$ some large sets get larger, and small sets stay small and do not even get new members. If λ cooperates, M will be a real model.

Our approach here is somewhat different. Our M_ζ is an incomplete type, or partial model, containing a large number of complete types which form a tree $\mathscr{T}$ under inclusion, in such a way that incompatible extensions in $\mathscr{T}^\zeta$ of a particular type p are allowed no further common variables. In this framework the index ζ runs only over $\operatorname{cof}\lambda$. The underlying set of the final model M will be λ^+. For $\alpha<\lambda^+$ the restriction of M to α is itself the limit of approximations M_α^ζ. A node p in $\mathscr{T}^\zeta$ describes M_α^ζ up to isomorphism, but a single node will correspond to as many as λ^+ distinct values of α (hence the variables in p will be systematically replaced by new variables for each suitable α). In a word, $\mathscr{T}^\zeta$ carries a number of templates describing various moderately large pieces of M.

4.2. *Notation*

Our goal in the present section is to show that $\square_\lambda^{b*}$ yields the completeness theorem for $\mathscr{L}(Q)$ in the λ^+-interpretation, and that $\square_\lambda^{a*}$ yields the corresponding omitting types theorem. To a large extent the two arguments may be given simultaneously.

We fix a λ^+-like ordering L, and an L-decomposition $\bar{C}=\langle C_a^\zeta : a\in L, \zeta<\operatorname{cof}\lambda\rangle$ as afforded by $\square_\lambda^{a*}$ or $\square_\lambda^{b*}$, as the case may be. We have an associated system E^ζ of equivalence relations on L satisfying certain conditions.

In either case we then define trees $\mathscr{T}^\zeta$ for $\zeta<\operatorname{cof}\lambda$ as follows. The nodes of $\mathscr{T}^\zeta$ are the classes a/E^ζ $(a\in L)$. Thus $|\mathscr{T}^\zeta|\leqq\lambda$, and if we are dealing with $\square_\lambda^{a*}$ then $|\mathscr{T}^\zeta|<\lambda$. The ordering on $\mathscr{T}^\zeta$ is defined as follows: $a/E^\zeta<b/E^\zeta$ if for some $a'\in a/E^\zeta$, $b'\in b/E^\zeta$, we have $a'\in C_{b'}^\zeta$. Observe that by $\square_\lambda^{b*}(1)$ there is then $a^*\in C_b^\zeta$ with $E^\zeta(a,a^*)$. Hence this relation is asymmetric (remembering (2) of $\square_\lambda^{b*}$) and transitive, and the predecessors of b/E^ζ are simply the classes a/E^ζ for $a\in C_b^\zeta$. Observe that if $a<a'$ and $a,a'\in C_b^\zeta$ then $a/E^\zeta<a'/E^\zeta$ in $\mathscr{T}^\zeta$, so $\mathscr{T}^\zeta$ really is a tree.

4.3. *Construction*

We now carry out a Henkin-style proof of the completeness theorem for $\mathscr{L}(Q)$ (assuming $\square_\lambda^{b*}$), or the omitting types theorem for $\mathscr{L}(Q)$ (assuming $\square_\lambda^{a*}$), in the λ^+-interpretation, using $\mathscr{T}^\zeta$-trees of types for $\zeta<\operatorname{cof}\lambda$.

Let T be a consistent $\mathscr{L}(Q)$-theory in a language τ of cardinality λ, and let p_i $(i<\lambda)$ be $\mathscr{L}(Q)$-types in the same language. Let $\tau=\bigcup_{\zeta<\operatorname{cof}\lambda}\tau^\zeta$ (an increasing union) with $|\tau^\zeta|<\lambda$. (When working with $\square_\lambda^{b*}$ we can allow $\tau^\zeta=\tau$ for all ζ, instead.) We will construct $(T,\mathscr{T}^\zeta)$-trees of types $\bar{p}^\zeta$ in the languages $\bar{\tau}^\zeta$ (extended by a suitable supply of free variables Z^ζ), together with embeddings $\iota_{tt'}^{\xi\zeta}:\bar{z}_t^\xi\to\bar{z}_{t'}^\zeta$ for $\xi\leqq\zeta$ whenever $t=t'/E^\xi$ (which means that t' has the form a/E^ζ for some a and $t=a/E^\xi$) so that for $a\in L$, if $t(\zeta)=a/E^\zeta$ then the family $(\bar{z}_{t(\zeta)}^\zeta;\iota_{t(\xi),t(\zeta)}^{\xi\zeta})$ forms a directed system. We proceed inductively for $\zeta<\operatorname{cof}\lambda$. Let $\lambda_\zeta=\max(|\tau^\zeta|,|\mathscr{T}^\zeta|)$. The conditions are as follows:

(0) $\tau^\zeta\subseteq\bar{\tau}^\zeta\subseteq\tau$, $|\bar{\tau}^\zeta|=\lambda_\zeta$;

(1) $\| \bar{p}^{\zeta} \| \leqq \lambda_{\zeta}$;
(2) each p_t^{ζ} is complete for $\tau^{\zeta}[Z_t^{\zeta}]$;
(3) each p_t^{ζ} has the properties described in Corollary 3.6(a,b,c) relative to the language τ^{ζ};
(4) (assuming $\square_{\lambda}^{a*}$) for $i < \lambda_{\zeta}$, $t \in \mathscr{T}^{\zeta}$, $z \in Z_t^{\zeta}$, there is $\varphi(z) \in p_i$ with $\neg\varphi(z) \in p_t^{\zeta}$;
(5) if $\xi < \zeta < \operatorname{cof} \lambda$ and $t = a/E^{\zeta} \in \mathscr{T}^{\zeta}$, let $t(\xi) = a/E^{\xi}$ (recall that E^{ζ} refines E^{ξ}); we require:

$$\iota_{t(\xi),t*}^{\xi\zeta}[p_{t(\xi)}^{\xi}] \subseteq p_t^{\zeta},$$

where the subscripted $*$ indicates the induced action on types.

To begin the construction for a given ζ, first let T^{ζ} be the deductive closure of T in the language τ^{ζ}. Let $q_t = (\bigcup_{\xi<\zeta} p_{t(\xi)}) \cup T^{\zeta}$. Applying 3.6 to $\bar{q}$, we obtain a $(T, \mathscr{T}^{\zeta})$-tree of types $\bar{q}^{\zeta}$ satisfying (0–3, 5). Assuming $\square_{\lambda}^{a*}$, each q_t^{ζ} is of cardinality at most λ_{ζ}. In order to treat (4) on the same footing as the other requirements we therefore need the following:

4.4. Lemma. *Let $\bar{p}$ be a $\mathscr{T}$-tree of types, $t \in \mathscr{T}$, $|p_t| < \lambda$, $z \in Z_t$. Suppose p is a type with no λ-support. Then there is a $\mathscr{T}$-tree $\bar{q}$ of types with $\bar{p} \subseteq \bar{q}$, $|\bar{q}| = |\bar{p}|$, and a formula $\varphi \in p$ with $\neg\varphi(z) \in q_t$.*

Proof. Combine Lemma 3.4 with the definition of λ-support. Note however that the notion of λ-support as defined here involves a well-ordered quantifier string, and we are allowing nonwellfounded trees. However, if we introduce a generalized notion of "linearly ordered" λ-supports, then the sets of finitary approximations to such generalized supports are equivalent to sets of finitary approximations to well-ordered λ-supports (using a well-ordering of the set of finite increasing sequences of variables in the generalized support). As it is only these sets of finite approximations which play a role in the argument, our claim follows. ■

4.5. *The model*

Let $\langle \bar{p}^{\zeta} : \zeta < \operatorname{cof} \lambda \rangle$ be the trees of types constructed in 4.3. For $\zeta < \operatorname{cof} \lambda$ and $a \in L$ let $\bar{z}_a^{\zeta} = \bar{x}_a^{\zeta} \bar{y}_a^{\zeta}$ be a new string of variables corresponding to the variables $\bar{z}_t^{\zeta}$ where $t = a/E^{\zeta}$. For $\xi \leqq \zeta$, let $\iota_a^{\xi\zeta} : \bar{z}_a^{\xi} \to \bar{z}_a^{\zeta}$ correspond to $\iota_{a/E^{\xi}, a/E^{\zeta}}^{\xi\zeta}$, and more generally if $\xi \leqq \zeta$, $a \in C_b^{\zeta}$, let $\iota_{ab}^{\xi\zeta}$ be the composition of $\iota_a^{\xi\zeta}$ with the inclusion from $\bar{z}_a^{\zeta}$ to $\bar{z}_b^{\zeta}$.

For $a \in L$, $t = a/E^{\zeta}$, $\zeta < \operatorname{cof} \lambda$, let q_a^{ζ} be

$$\{\varphi(\cdots \bar{z}_b^{\zeta} \cdots)_{b \in \{a\} \cup C_a^{\zeta}} : \varphi(\cdots \bar{z}_s^{\zeta} \cdots)_{s \leqq t} \in p_t^{\zeta}\}.$$

Then q_a^ζ is a specific alphabetical variant of p_t^ζ. Let $q = \varinjlim \langle q_a^\zeta : a \in L, \zeta < \operatorname{cof} \lambda \rangle$ where the direct limit is taken over $\operatorname{cof} \lambda \times L$, with respect to the maps $\iota_{ab*}^{\xi\zeta} : q_a^\xi \to q_b^\zeta$ induced by $\langle \iota_{ab}^{\xi\zeta} : \xi \leqq \zeta, a \in C_b^\zeta \rangle$. For simplicity we will henceforth treat these maps notationally as inclusion maps. Then:

(1) q is closed under conjunction.

Let $\varphi \in q_a^\zeta$, $\psi \in q_b^\xi$. Without loss of generality $a \leqq b$. Choose $\rho \geqq \zeta, \xi$ so that $a \in C_b^\rho$. Then $\varphi, \psi \in q_b^\rho$ and hence $\varphi \,\&\, \psi \in q_b^\rho$.

(2) q is consistent.

As each q_a^ζ is consistent, this follows from (1).

(3) q is complete.

As each q_a^ζ is complete in the language $\tau^\zeta[Z_a^\zeta]$ $(Z_a = \{\bar{z}_b^\zeta : b \in \{a\} \cup C_a^\zeta\})$, it suffices to note that for any formula φ of the language τ in the variables $\bar{z}_i = \bar{z}_{a(i)}^{\zeta(i)}$, if $a = \sup\{a(i)\}$ and $\zeta_0 = \sup(\zeta(i))$, there is $\zeta \geqq \zeta_0$ with $\varphi \in \tau^\zeta$ so that each $a(i)$ is in $\{a\} \cup C_a^\zeta$, and then φ or $\neg\varphi$ will be in q_a^ζ. In particular:

(4) The atomic part of q defines a structure M.

It remains to check that q describes a correct Henkin construction.

4.6. Lemma. *q is the complete $\mathscr{L}(Q)$-diagram of M.*

Proof. We show by induction that for any $\varphi(\bar{z})$ with suitable free variables (treated as constants representing elements of M):

$$(*) \qquad \varphi \in q \quad \text{iff} \quad M \models \varphi.$$

As negation takes care of itself and the atomic and conjunctive cases were handled implicitly in (1–4) above, we confine our attention to the two quantifiers $\exists$, Q, and the question of omitting types. According to 4.3(3,4) $\exists$ and the omitting types problem (assuming $\square_\lambda^{a*}$) have been dealt with properly. It remains to check that the part of 4.3(3) corresponding to 3.6(b,c) provides an adequate treatment of the quantifier Q.

If $Qu\,\varphi(u, \bar{z}) \in q$, more specifically $Qu\,\varphi(u, \bar{z}) \in q_a^\zeta$, then for $a < b \in L$ and large ζ we will have $\varphi(x_b, \bar{z})$ for some $x_b \in \bar{x}_b^\zeta$. For $b < c$ we will also have $x_b \neq x_c \in q$ (else we get $Qx(x = x_b)$ in some q_b^ζ). Thus $Qu\,\varphi(u, \bar{z})$ will hold in M.

Suppose now that $Qu\,\varphi(u, \bar{z}) \notin q$, so $\neg Qu\,\varphi(u, \bar{z}) \in q$, specifically $\neg Qu\,\varphi(u, \bar{z}) \in q_a^\zeta$. By 4.5(1) and the part of 4.3(3) corresponding to 3.6(c), if

$\varphi(z, \bar{z})$ holds in M then z has a name z' in $\bar{y}_a^\zeta$ for large ζ. Thus there are at most λ such elements in M, as desired. ■

§5. Getting to square eight

5.1. PROPOSITION. *Suppose that* $(\aleph_0, \aleph_1) \xrightarrow[\aleph_0]{} (\lambda, \lambda^+)$. *Then* $\square_\lambda^{b*}$ *holds.*

PROOF.

A. We first show that there is a model M (in a c.c.c. extension of the universe of set theory) with universe ω_1 equipped with relations $<$, P, Q, R and functions F, G_i $(i = 1, 2)$, H, I, J satisfying:

1. $<$ is the usual well ordering, P is a predicate picking out ω.
2. Q is a predicate distinguishing an unbounded subset of ω, not containing 0.
3. F is a partial 2-place function on M defined for (α, β) with $\omega \leqq \alpha < \beta$; we write $F_\beta(\alpha)$ instead of $F(\alpha, \beta)$ and we assume that $F_\beta: [\omega, \beta) \xrightarrow{1\text{-}1} \omega - Q$.
4. R is a binary relation; $R(n, \alpha)$ implies $n < \omega \leqq \alpha < \omega_1$; we write R_α for the set $\{n : R(n, \alpha)\}$; and we require that the sets R_α are unbounded in ω and almost disjoint.
5. J is a 2-place function from $[\omega, \omega_1)$ into ω, and for each $\beta \in [\omega, \omega_1)$, the sets $R_\alpha \cap (J(\alpha, \beta), \omega)$ (α varies over $[\omega, \beta)$) are pairwise disjoint.
6. For $\alpha < \beta < \gamma$ in $\omega_1 - \omega$, and $n \in Q$, if $F_\gamma(\beta) < n$ then:
$$F_\beta(\alpha) < n \quad \text{iff} \quad F_\gamma(\alpha) < n.$$
7. H is a partial 2-place function on M defined for (β, n) with $n \in Q$, $\beta \in [\omega, \omega_1)$; we write $H_n(\beta)$ for $H(\beta, n)$, and we require that for $n \in Q$, $\beta \in [\omega, \omega_1)$, we have
$$n > H_n(\beta) > \sup\{m \in Q : m < n\}.$$
8. I is a partial 4-place function defined for $(n, \beta, \gamma, \alpha)$ with $n \in Q$, $\omega \leqq \alpha \leqq \beta < \omega_1$, $\gamma \in [\omega, \omega_1)$ if $H_n(\beta) = H_n(\gamma)$ and either $F_\beta(\alpha) < n$ or $\alpha = \beta$. If $n \in Q$, $\beta, \gamma \in [\omega, \omega_1)$, and $H_n(\beta) = H_n(\gamma)$, then $I(n, \beta, \gamma; -)$ is a 1–1 order-preserving function from $\{\alpha \in [\omega, \beta) : F_\beta(\alpha) < n\} \cup \{\beta\}$ onto $\{\alpha \in [\omega, \beta) : F_\gamma(\alpha) < n\} \cup \{\gamma\}$ which preserves the values of $F_{\alpha_1}(\alpha_2)$, $H_m(\alpha_2)$ for $m \leqq n$, $m \in Q$.
9. G_i $(i = 1, 2)$ are partial two-place functions from ω to ω; if $n \in Q$, $\omega \leqq \alpha < \beta < \omega_1$, $F_\beta(\alpha) < n$, and $m_1 < m_2$ are in Q, then we have:
   (a) $H_{m_1}(\alpha) = G_2(H_{m_2}(\alpha), m_1)$;
   (b) $F_\beta(\alpha) = G_1(H_n(\beta), H_n(\alpha))$;
   (c) $H_n(\beta) \neq H_n(\alpha)$.

PROOF OF THE CLAIM. We can choose $<$, P satisfying (1). By an *approximation* to Q, F, H, G_i $(i = 1, 2)$, I we mean a 7-tuple $p = (u, q, f, h, g_1, g_2, i)$ such that u is a finite subset of ω_1, $u \cap \omega$ is an initial segment of ω, $\max\{u \cap \omega\} \in q$, and the analogues of conditions (3, 6–9) hold on u. The components of p will be denoted u^p, q^p, etc. We write $p \leqq r$ if $u^p \subseteq u^r$ and the remaining components of p are restrictions of their counterparts in r. Let $\mathscr{P}$ be the partially ordered set of all approximations. Then $\mathscr{P}$ satisfies the countable chain condition, as one may check, and for each $i < \omega_1$ the set D_i of approximations p for which $i \in u^p$ is dense.

A $\mathscr{P}$-generic set encodes a model satisfying (1–3, 6–9); now define R so that (4) holds, and then define J so that (5) holds. As we are only interested in those aspects of the situation which can be encoded in $L(Q)$, a similar model exists absolutely. For a more "direct" description of the model (that is, without first forcing) compare [Sh3, Lemma 13].

B. Let ψ be a sentence in $L(Q)$ expressing the properties (1–9) of M. Take a model $N \models \psi$ with $\| N \| = \lambda^+$, $|P^N| = \lambda$. Let $L = N - P^N$. We now claim:

(i) L is λ^+-like (by (3) initial segments have cardinality at most λ);

(ii) $\operatorname{cof}(P^N, < \restriction_{P^N}) = \operatorname{cof} \lambda$.

Suppose on the contrary that $\kappa = \operatorname{cof} P \neq \operatorname{cof} \lambda$. We can write P as the increasing union of subsets P_ζ ($\zeta < \operatorname{cof} \lambda$), each of cardinality less than λ. For $a \in L$ let $R_a = \{x \in P^N : R(x, a)\}$. Fix a subset A_a of R_a of order type κ, unbounded in P. For each $a \in L$ fix $\zeta(a)$ with $|A_a \cap P_{\zeta(a)}| = \kappa$. Fix $\zeta < \operatorname{cof} \lambda$ so that the set $B = \{a \in L : \zeta(a) = \zeta\}$ is unbounded in L. For $a \in B$ let A'_a be $A_a \cap P_\zeta$. Choose $b \in L$ so that the set $B_0 = \{a \in B : a < b\}$ has cardinality λ. For $a \in B_0$ let A''_a be $\{i \in A'_a : i > J(a, b)\}$. Then the sets A''_a $(a \in B_0)$ form a collection of λ disjoint nonempty subsets of P_ζ, a contradiction to "$|P_\zeta| < \lambda$".

REMARK. If λ is a singular strong limit cardinal there is a simpler argument based on condition (5). If $P = \bigcup_{\zeta < \operatorname{cof} \lambda} P_\zeta$ with $|P_\zeta| < \lambda$, and $\operatorname{cof}(P, <) \neq \operatorname{cof} \lambda$, then for each $a \in L$, there is $\zeta_a < \operatorname{cof} \lambda$ with $R_a \cap P_{\zeta_a}$ unbounded in P, hence for some ζ we have $|\{a : \zeta_a = \zeta\}| = \lambda^+$, contradicting $2^{|P_\zeta|} < \lambda$ (this type of argument was first used by Litman).

C. (In the proof of Proposition 5.2 an additional step will be inserted at this point.)

D. Fix an increasing cofinal sequence $\langle n_\zeta : \zeta < \operatorname{cof} \lambda \rangle$ in P. For $\zeta < \operatorname{cof} \lambda$, $b \in L$, let C^ζ_b be $\{a \in L : a < b \text{ and } F_b(a) < n_\zeta\}$. This is an L-decomposition; coherence follows from condition (6).

To verify $\square_\lambda^{b*}$, it remains to introduce a suitable equivalence relation. For $\zeta < \operatorname{cof}\lambda$, and $n = n_\zeta$, let $E^\zeta(a, b)$ hold for $a, b \in L$ iff $H_n(a) = H_n(b)$. To see that E^ξ refines E^ζ for $\zeta < \xi$, use (9a). Furthermore each E^ζ has at most λ classes since the range of H_n is bounded by n. We have more points to verify:

(iii) If $b \in C_c^\xi$ then $E^\zeta(b, c)$.
This follows from (9c) and (3).

(iv) If $E^\zeta(c_1, c_2)$ holds and $b_1 \in C_{c_1}^\zeta$ then for some $b_2 \in C_{c_2}^\zeta$, $E^\zeta(b_1, b_2)$ holds.
Notice that in this case $b_2 = I(n_\zeta, c_1, c_2\, b_1)$ is defined. By (8), $H_{n_\zeta}(b_2) = H_{n_\zeta}(b_1)$. Also by (8), $F_{c_2}(b_2) = F_{c_1}(b_1) < n_\zeta$, so $b_2 \in C_{c_2}^\zeta$. ■

(v) $\mathscr{C}_a^\zeta$, $\mathscr{C}_b^\zeta$ if $E^\zeta(a, b)$.
Again use $I(n_\zeta, a, b, -)$.

5.2. Proposition. *Suppose that* $(\aleph_0, \aleph_1) \xrightarrow[\aleph_0]{} (\lambda, \lambda^+)$ *and* $2^{\operatorname{cof}\lambda} < \lambda$. *Then* $\square_\lambda^{a*}$ *holds.*

Proof. A, B. We proceed as in the proof of the previous proposition. Build a model M by forcing, as before, having one additional function G_0 subject to one further constraint in the context of condition (9) above:

9d. $H_n(\alpha) = G_0(H_n(\beta), F_\beta(\alpha))$,

and in addition:

10. For $m, n < \omega$, $\inf\{k \in Q : k \geqq m, n\} \geqq G_i(m, n)$ for $i = 0, 1, 2$.

Then by absoluteness and the assumed two-cardinal transfer property, we get a model N of the $L(Q)$-content of these properties, in the λ^+-interpretation. In this model there is an initial segment P of cofinality $\operatorname{cof}\lambda$, and a terminal segment L equipped with a λ^+-like ordering $<$.

Write P as the increasing union of bounded subsets P_ζ ($\zeta < \operatorname{cof}\lambda$) of cardinality less than λ. We may suppose that $P_0 = \varnothing$, that each P_ζ has a maximum element n_ζ, which belongs to Q, and (applying condition (10)) that each P_ζ is closed under the functions G_i ($i = 0, 1, 2$).

C. Assume now that $2^{\operatorname{cof}\lambda} < \lambda$. Then we claim that, without loss of generality, $H_{n_\zeta}(a) \in P_\zeta$ for $\zeta < \operatorname{cof}\lambda$, $a \in L$.

For $a \in L$ and $\zeta < \operatorname{cof}\lambda$, choose $\xi_a(\zeta) < \operatorname{cof}\lambda$ with $H_{n_\zeta}(a) \in P_{\xi_a(\zeta)}$; we may take ξ_a to be increasing in ζ. For $\xi : \operatorname{cof}\lambda \to \operatorname{cof}\lambda$, let B_ξ be $\{a \in L : \xi_a = \xi\}$, and choose ξ so that B_ξ is unbounded in L. Replace Q by the sequence (n_ζ), replace L by $L_\xi = \{a \in L$: for all $\zeta < \operatorname{cof}\lambda$, $H_{n_\zeta}(a) \in P_{\xi(\zeta)}\}$, and replace the sequence $\langle P_\zeta : \zeta < \operatorname{cof}\lambda\rangle$ by $\langle P'_\zeta : \zeta \in B_\xi\rangle$ where $P'_\zeta = \{b \in P_{\xi(\zeta)} : b \leqq n_\zeta\}$.

$L_\xi \supseteq B_\xi$ is unbounded in L. It is necessary to check that L_ξ is closed under the action of I. So let $a, b, c \in L_\xi$ with $a \leqq b$, and $\zeta < \operatorname{cof} \lambda$, with $H_{n_\zeta}(b) = H_{n_\zeta}(c)$, and $F_b(a) < n_\zeta$ (since the case $a = b$ is trivial), and let $a' = I(n_\zeta, b, c, a)$. Let $\zeta' < \operatorname{cof} \lambda$, $n = n_{\zeta'}$. We claim that $H_n(a') \in P_{\xi(\zeta')}$. If $\zeta' \leqq \zeta$ then, by (8), $H_n(a') = H_n(a)$; so suppose that $\zeta' > \zeta$. Since $a \in L_\xi$, by (9d) and the closure condition on $P_{\xi(\zeta')}$ it suffices to check that $F_c(a') \in P_{\xi(\zeta')}$; as $F_c(a') = F_b(a)$ this will follow from (9b).

D. For $\zeta < \operatorname{cof} \lambda$, $b \in L$, let C_b^ζ be $\{a \in L : a < b \text{ and } F_b(a) \in P_\zeta\}$. We claim that this is an L-decomposition; we must check the coherence. Accordingly fix $a < b < c$ in L with $b \in C_c^\zeta$ and assume $a \in C_b^\zeta \cup C_c^\zeta$; then by (6), $F_b(a)$, $F_c(a) < n_\zeta$, and by (9b) and the closure condition on P_ζ, $F_b(a)$, $F_c(a) \in P_\zeta$, as required. By (3), $|C_b^\zeta| \leqq |P_\zeta| < \lambda$.

The equivalence relations E^ζ are defined as in the proof of Proposition 5.1 above. By our present construction, each E^ζ has fewer than λ equivalence classes. The rest of the argument is as in the previous case. ∎

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